Notes 4 Appendix 2

Asymptotic Result on $\sum_{n\le x} d_k(n)$ for $k \ge 2$.

The main result of this section is

Theorem 1 *For* $k \ge 2$

$$\sum_{n\le x} d_k(n) = xP_{k-1}(\log x) + O\left(x^{1-1/k}\log^{k-2} x\right), \tag{1}$$

where $P_d(y)$ *is a polynomial of degree* d *in* y*, with leading coefficient* $1/d!$.

The proof of this proceeds with two lemmas.

Lemma 2 *For all integers* $\ell \ge 0$ *there exists constants* C_ℓ *such that*

$$\sum_{n\le x} \frac{\log^\ell n}{n} = \frac{1}{\ell+1}\log^{\ell+1} x + C_\ell + O\left(\frac{\log^\ell x}{x}\right)$$

for $x > x_0(\ell)$.

Note this generalises the $\ell = 0$ case seen in the notes when C_0 is Euler's constant.

Proof By partial summation

$$\sum_{n\le x} \frac{\log^\ell n}{n} = \frac{1}{x}\sum_{n\le x}\log^\ell n + \int_1^x \sum_{n\le t}\log^\ell n \frac{dt}{t^2}. \tag{2}$$

We estimate $\sum_{n\le t}\log^\ell n$ by replacing it by an integral

$$\sum_{n\le t}\log^\ell n = \int_1^t \log^\ell y dy + O\left(\log^\ell t\right), \tag{3}$$

when repeated integration by parts gives a main term of $tQ_\ell(\log t)$ for some polynomial of degree ℓ, though this is not required here. Instead, substituting (3) into (2) gives

$$\sum_{n\le x} \frac{\log^\ell n}{n} = \frac{1}{x}\int_1^x \log^\ell y dy + O\left(\frac{\log^\ell x}{x}\right) + \int_1^x \left(\int_1^t \log^\ell y dy + \varepsilon_\ell(t)\right)\frac{dt}{t^2}, \tag{4}$$

where $\varepsilon_\ell(t) \ll \log^\ell t$. In the double integral of $\log^\ell y$, interchange the integrals to get

$$\int_1^x \log^\ell y \left(\int_y^x \frac{dt}{t^2} \right) dy = \int_1^x \log^\ell y \left(\frac{1}{y} - \frac{1}{x} \right) dy = \frac{\log^{\ell+1} x}{l+1} - \frac{1}{x} \int_1^x \log^\ell y dy.$$

The integral here cancels the first in (4).

All that remains is to estimate the contribution from the error $\varepsilon_\ell(t)$ within the integral in (4). Because $\varepsilon_\ell(t) \ll \log^\ell t$, the integral converges and can be replaced by the integral over $[1, \infty)$,

$$\int_1^x \varepsilon_\ell(t) \frac{dt}{t^2} = \int_1^\infty \varepsilon_\ell(t) \frac{dt}{t^2} - \int_x^\infty \varepsilon_\ell(t) \frac{dt}{t^2},$$

and this first integral is the constant C_ℓ. For the tail end we have

$$\int_x^\infty \varepsilon_\ell(t) \frac{dt}{t^2} \ll \int_x^\infty \log^\ell t \frac{dt}{t^2} \ll \frac{\log^\ell x}{x^2},$$

by a question on Problem Sheet 3.

Combining we get the stated result. ■

Lemma 3

$$\sum_{a \le U} \frac{\log^r(x/a)}{a} = \int_{x/U}^x \frac{\log^r t}{t} dt + \sum_{\ell=0}^r (-1)^\ell \binom{r}{\ell} C_\ell \log^{r-\ell} x + O\left(\frac{\log^r x}{U} \right),$$

for $U \ge U_0(r)$.

Proof Use the binomial expansion

$$\log^r(x/a) = (\log x - \log a)^r = \sum_{\ell=0}^r (-1)^\ell \binom{r}{\ell} (\log x)^{r-\ell} (\log a)^\ell.$$

For then

$$\sum_{a \le U} \frac{\log^r(x/a)}{a} = \sum_{\ell=0}^r (-1)^\ell \binom{r}{\ell} (\log x)^{r-\ell} \sum_{a \le U} \frac{\log^\ell a}{a} \tag{5}$$

$$= \sum_{\ell=0}^r (-1)^\ell \binom{r}{\ell} (\log x)^{r-\ell} \left(\frac{1}{\ell+1} \log^{\ell+1} U + C_\ell + O\left(\frac{\log^\ell U}{U} \right) \right),$$

for $U \geq U_0(r)$, by Lemma 2. Note that

$$\begin{aligned}
\binom{r}{\ell}\frac{1}{\ell+1} &= \frac{r!}{\ell!\,(\ell+1)\,(r-\ell)!} \\
&= \frac{(r+1)!}{(\ell+1)!\,((r+1)-(\ell+1))!\,(r+1)} \\
&= \binom{r+1}{\ell+1}\frac{1}{r+1}.
\end{aligned}$$

For any α and β we have

$$\begin{aligned}
\sum_{\ell=0}^{r}(-1)^{\ell}\binom{r}{\ell}\frac{1}{\ell+1}\alpha^{r-\ell}\beta^{\ell+1} &= \sum_{\ell=0}^{r}(-1)^{\ell}\binom{r+1}{\ell+1}\frac{1}{r+1}\alpha^{r-\ell}\beta^{\ell+1} \\
&= \frac{-1}{r+1}\sum_{\ell=1}^{r+1}(-1)^{\ell}\binom{r+1}{\ell}\alpha^{(r+1)-\ell}\beta^{\ell} \\
&= \frac{1}{r+1}\left(\alpha^{r+1}-(\alpha-\beta)^{r+1}\right),
\end{aligned}$$

by the Binomial Theorem again. Applied within (5) with $\alpha = \log x$ and $\beta = \log U$ this gives

$$\begin{aligned}
\sum_{\ell=0}^{r}(-1)^{\ell}\binom{r}{\ell}\frac{1}{\ell+1}(\log x)^{r-\ell}\log^{\ell+1}U &= \frac{1}{r+1}\left(\log^{r+1}x-(\log x-\log U)^{r+1}\right) \\
&= \frac{1}{r+1}\left(\log^{r+1}x-\log^{r+1}(x/U)\right) \\
&= \int_{x/U}^{x}\frac{\log^{r}t}{t}dt,
\end{aligned}$$

which gives the stated result. ■

Note that a change of variable gives

$$\int_{x/U}^{x}\frac{\log^{r}t}{t}dt = \int_{1}^{U}\frac{\log^{r}(x/w)}{w}dw,$$

and this integral has a form closer to that of the sum it is approximating than the integral shown.

Proof of Theorem by induction. When $k = 2$ it has been shown in the notes that

$$\sum_{n\le x} d(n) = x\log x + (2\gamma - 1)\,x + O\big(x^{1/2}\big) \tag{6}$$

which agrees with (1) in this case.

Assume the result holds for d_k for some $k \ge 3$.

For the d_{k+1} case apply the Hyperbolic Method as

$$\begin{aligned}\sum_{n\le x} d_{k+1}(n) &= \sum_{n\le x} 1 * d_k(n)\\ &= \sum_{a\le U}\sum_{b\le x/a} d_k(b) + \sum_{b\le V} d_k(b)\sum_{a\le x/b} 1 - [U]\sum_{b\le V} d_k(b)\,, \end{aligned} \tag{7}$$

with U and V to be chosen to minimise the error terms subject to $UV = x$.

First term in (7) .

For the first sum in (7) we apply the inductive hypothesis as

$$\sum_{a\le U}\sum_{b\le x/a} d_k(b) = \sum_{a\le U}\frac{x}{a}P_{k-1}\left(\log\frac{x}{a}\right) + O\left(\sum_{a\le U}\left(\frac{x}{a}\right)^{1-1/k}\log^{k-2}x\right). \tag{8}$$

Write $P_{k-1}(y) = \sum_{r=0}^{k-1} c_{k-1,r}y^r$ for some coefficients $c_{k-1,r}$. Then, by Lemma 3,

$$\begin{aligned}\sum_{a\le U}\frac{x}{a}P_{k-1}\left(\log\frac{x}{a}\right) &= x\sum_{r=0}^{k-1} c_{k-1,r}\sum_{a\le U}\frac{\log^r(x/a)}{a}\\ &= x\sum_{r=0}^{k-1} c_{k-1,r}\left(\int_{x/U}^{x}\frac{\log^r t}{t}dt + \sum_{\ell=0}^{r}(-1)^\ell\binom{r}{\ell}C_\ell\log^{r-\ell}x\right)\\ &\qquad O\left(x\sum_{r=0}^{k-1}|c_{k-1,r}|\,\frac{\log^r x}{U}\right)\end{aligned} \tag{9}$$

Second term in(7) .

The second sum is

$$\sum_{b\le V} d_k(b)\sum_{a\le x/b} 1 = x\sum_{b\le V}\frac{d_k(b)}{b} + O\left(\sum_{b\le V} d_k(b)\right). \tag{10}$$

This error here is $O(V\log^{k-1}V)$ from the inductive hypothesis. For the other term in (10) use summation by parts as

$$\begin{aligned}\sum_{b\le V}\frac{d_k(b)}{b} &= \frac{1}{V}\sum_{b\le V}d_k(b)+\int_1^V\sum_{b\le t}d_k(b)\,\frac{dt}{t^2}\\ &= P_{k-1}(\log V)+O\big(V^{-1/k}\log^{k-1}V\big) \qquad (11)\\ &\qquad +\int_1^V\big(tP_{k-1}(\log t)+\eta_{k-1}(t)\big)\,\frac{dt}{t^2},\end{aligned}$$

where $\eta_{k-1}(t)\ll t^{1-1/k}\log^{k-1}t$, again using the inductive hypothesis. The integral over this error converges and so we complete it to infinity and bound the tail end:

$$\int_1^V\eta_{k-1}(t)\,\frac{dt}{t^2} = \int_1^\infty\eta_{k-1}(t)\,\frac{dt}{t^2}-\int_V^\infty\eta_{k-1}(t)\,\frac{dt}{t^2},$$

say. And

$$\int_V^\infty\eta_{k-1}(t)\,\frac{dt}{t^2} \ll \int_V^\infty t^{1-1/k}\log^{k-1}t\,\frac{dt}{t^2} \ll \frac{\log^{k-1}V}{V^{1/k}}. \qquad (12)$$

Third term in (7).

$$[U]\sum_{b\le V}d_k(b)=(U+O(1))\left(VP_{k-1}(\log V)+O\big(V^{1-1/k}\log^{k-2}V\big)\right),$$

by the inductive Hypothesis.

Error terms in (7).

The errors from the first term are $O\big(x^{1-1/k}U^{1/k}\log^{k-2}x\big)$ from (8) and $O\big(xU^{-1}\log^{k-1}x\big)$ from (9).

The errors from the second term are $O\big(V\log^{k-1}V\big)$ from (10) and $O\big(xV^{-1/k}\log^{k-1}V\big)$ from both (11) and (12).

The errors from the third term are $O\big(V\log^{k-1}x\big)$ and $O\big(UV^{1-1/k}\log^{k-2}x\big)$.

It is easy to check that, because $UV=x$, we only have two independent errors, $O\big(xU^{-1}\log^{k-1}x\big)$ and $O\big(xV^{-1/k}\log^{k-1}V\big)$.

We minimise the errors by equating these two, i.e. $U^{-1}=V^{-1/k}$, that is $V=U^k$. With $UV=x$ this means $U=x^{1/(k+1)}$ and $V=x^{k/(k+1)}$.

Then the overall error in (7) is $O\big(x^{k/(k+1)}\log^{k-1}x\big)$

Main Terms in (7) .

We have seen above the main term of the first sum in (8) ,

$$x\sum_{r=0}^{k-1}c_{k-1,r}\left(\int_{x/U}^{x}\frac{\log^{r}t}{t}dt+\sum_{\ell=0}^{r}(-1)^{\ell}\binom{r}{\ell}C_{\ell}\log^{r-\ell}x\right)=$$

$$=x\int_{x/U}^{x}P_{k-1}(\log t)\,\frac{dt}{t}+x\sum_{r=0}^{k-1}c_{k-1,r}\sum_{\ell=0}^{r}(-1)^{\ell}\binom{r}{\ell}C_{\ell}(\log x)^{r-\ell}.$$

We have seen above the main term of the second sum in (8) ,

$$xP_{k-1}(\log V)+x\int_{1}^{V}tP_{k-1}(\log t)\,\frac{dt}{t^{2}}+B,$$

where $B=\int_1^\infty \eta_{k-1}(t)\,t^{-2}dt$.

And the main term of the third sum in (8) is

$$UVP_{k-1}(\log V)=xP_{k-1}(\log V)\,.$$

Add and subtract these to find the main term of $\sum_{n\le x}d_{k+1}(n)$ to be

$$x\int_{1}^{x}P_{k-1}(\log t)\,\frac{dt}{t}+x\sum_{r=0}^{k-1}c_{k-1,r}\sum_{\ell=0}^{r}(-1)^{\ell}\binom{r}{\ell}C_{\ell}(\log x)^{r-\ell}+B.$$

Complicated as it might appear, but this is $x\times$ polynomial in $\log x$. We take it to be the definition of $xP_k(\log x)$. Hence we have shown that

$$\sum_{n\le x}d_{k+1}(n)=xP_k(\log x)+O\big(x^{1-1/(k+1)}\log^{k-1}x\big)\,,$$

that is, our result holds for $k+1$. Hence, by induction, it holds for all $k\geq 2$.

Note that

$$\int_{1}^{x}P_{k-1}(\log t)\,\frac{dt}{t}\;=\;\sum_{r=0}^{k-1}\frac{c_{k-1,r}}{r+1}\log^{r+1}x$$

and it is from here that we see that the leading coefficient, $c_{k,k}$ in $P_k(\log x)$ satisfies

$$c_{k,k}=\frac{c_{k-1,k-1}}{k}$$

where $c_{k-1,k-1}$ is the leading coefficient in $P_{k-1}(\log x)$. Continuing

$$c_{k,k}=\frac{c_{1,1}}{k!}=\frac{1}{k!}$$

since the leading coefficient in (6), i.e. $c_{1,1}$ equals 1. ■